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Differentiation with respect to covariant coordinates.

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DIFFERENTIATION WITH RESPECT TO COVARIANT COORDINATES

LEONARD H. HARRIS

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THESIS

DIFFERENTIATION WITH RESPECT TO COVARIANT COORDINATES

Leonard H Harris

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DIFFERENTIATION WITH RESPECT TO COVARIANT COORDINATES

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Leonard H. Harris

DIFFERENTIATION WITH RESPECT TO COVARIANT COORDINATES

by

Leonard H. Harris
//
Captain, United States Army

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE

United States Naval Postgraduate School
Monterey, California

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DIFFERENTIATION WITH RESPECT TO COVARIANT COORDINATES

by

Leonard H. Harris

This work is accepted as fulfilling
the thesis requirements for the degree of
MASTER OF SCIENCE
from the
United States Naval Postgraduate School

ABSTRACT

There are systems in which differentiation with respect to covariant coordinates is a meaningful concept.

In these systems the covariant and contravariant components are related by the usual formula

a) $$dx_i = g_{ij}\,dx^j \, .$$

In order that the covariant components be coordinates, the metric tensors of the contravariant coordinates must satisfy the following relationships:

ORTHOGONAL SYSTEMS

b) $$g_{ii} = f_i(x^i) \qquad \text{no sum on } i;$$

NON-ORTHOGONAL SYSTEMS

c) $$\frac{\partial g_{im}}{\partial x^n} = \frac{\partial g_{in}}{\partial x^m} \qquad \text{all } i, m, n \, .$$

For a scalar point function Φ (which has, of course, a representation $\Phi_1(x_1, x_2, \ldots)$ in terms of the covariant components and a representation $\Phi_2(x^1, x^2, \ldots)$ in terms of the contravariant components) the equation linking the derivative with respect to the covariant coordinates and the derivative with respect to the contravariant coordinates is

d) $$\frac{\partial \Phi}{\partial x^i} = g_{ji}\frac{\partial \Phi}{\partial x_j} \, .$$

For tensor components related by the equation

e) $$V_i = g_{ij} V^j$$

this relationship becomes

f) $$\frac{\partial V_\alpha}{\partial x^i} = g_{ji}\frac{\partial V_\alpha}{\partial x_j} \, .$$

In order for d) and f) to be valid, the relationship between the covariant and contravariant coordinates must satisfy the system of

partial differential equations

g) $$\frac{\partial x_j}{\partial x^i} = g_{ji}$$

The writer wishes to express his appreciation for the assistance and encouragement given him by Professor Charles C. Torrance of the U. S. Naval Postgraduate School in this investigation.

In a recent paper (Cf #8 in the Bibliography), Craig introduces the notion of the derivative of a scalar point function with respect to a *covariant* variable. Craig's notation does not involve differentiation with respect to a covariant *coordinate*, since he defines his covariant variable not as a coordinate in itself, but as the derivative of a space coordinate with respect to a contravariant coordinate. Actually, Craig's space coordinate is a function of his set of *contravariant* coordinates. Indeed, he introduces differentiation with respect to the covariant variable merely to obtain a partial derivative which is contravariant. In this respect his derivative is similar in form to the contravariant derivative of a scalar point function.

Nonetheless, it appears that a derivative with respect to a genuine covariant *coordinate* would be of some usefulness, particularly in the development of the thoery of extensors.

The standard definition of differentiation involves division of the increment of some function by an increment of some variable. Whenever this variable is a *coordinate* the resulting derivative has the usual meaning of rate of change of the function with respect to a *displacement*. The coordinates with which we customarily deal are contravariant coordinates, since these are directly interpretable as components of a displacement. There are certain systems, however, in which *covariant* coordinates are meaningful quantities, as will be shown later. Let us assume a scalar point function $\mathcal{Q}$ in such a system. This function has a certain representation, say $\mathcal{Q}_1(x_1, x_2, \ldots)$ with respect to the covariant components of the system, and another representation, say $\mathcal{Q}_2(x^1, x^2, \ldots)$ with respect to the contravariant components. We will call both representations $\mathcal{Q}$. The relationship between the covariant and contravariant components is the usual one:

1) $$dx_i = g_{ij} \, dx^j .$$

We know that the covariant derivative of a scalar point function is defined as

2) $$\frac{\partial Q}{\partial x^i} = Q_{,i} \quad ,$$

and we also know that such a derivative yields the covariant components of a tensor. It thus appears desirable from the standpoint of duality to define the derivative with respect to the covariant coordinates in such a way that it will not only transform as a contravariant tensor, but will, in fact, be the contravariant derivative. In other words, it should satisfy the relationship

3) $$\frac{\partial Q}{\partial x^i} = g_{ji}\frac{\partial Q}{\partial x_j} \quad ,$$

where the g_{ji} represent the covariant components of the metric tensor of the set of contravariant coordinates.

To show that this is true, let us multiply both sides of 3) by $g^{\beta i}$ to get

4) $$g^{\beta i}\frac{\partial Q}{\partial x^i} = g^{\beta i} g_{ji}\frac{\partial Q}{\partial x_j} = \delta^{\beta}_{j}\frac{\partial Q}{\partial x_j} = \frac{\partial Q}{\partial x_\beta} .$$

If we substitute 2) into 4) it reduces to

5) $$\frac{\partial Q}{\partial x_\beta} = g^{\beta i}\frac{\partial Q}{\partial x^i} = g^{\beta i} Q_{,i} = Q^{,\beta} \quad ,$$

or, changing notation slightly,

6) $$\frac{\partial Q}{\partial x_j} = Q^{,j} \quad ,$$

so that $\frac{\partial Q}{\partial x_j}$ is indeed the contravariant derivative of the scalar point function Q .

For tensor components related by

7) $$V_i = g_{ij} V^j$$

the relationship corresponding to 3) is

8) $$\frac{\partial V_\alpha}{\partial x^i} = g_{ji}\frac{\partial V_\alpha}{\partial x_j} \quad ,$$

and that corresponding to 4) is

9) $$\frac{\partial V_\alpha}{\partial x_\beta} = g^{\beta i}\frac{\partial V_\alpha}{\partial x^i} \quad ,$$

although $\frac{\partial V_\alpha}{\partial x_\beta}$ is not, of course, the contravariant derivative. However, the contravariant derivative can be obtained from the definition of the covariant derivative

10) $$V_{\alpha,i} = \frac{\partial V_\alpha}{\partial x^i} - V_r \begin{Bmatrix} r \\ \alpha\, i \end{Bmatrix}$$

by multiplying both sides by $g^{\beta i}$ and using 9). Thus

11) $$\begin{aligned} V_\alpha{}^{,\beta} = g^{\beta i} V_{\alpha,i} &= g^{\beta i}\frac{\partial V_\alpha}{\partial x^i} - g^{\beta i} V_r \begin{Bmatrix} r \\ \alpha\, i \end{Bmatrix} \\ &= \frac{\partial V_\alpha}{\partial x_\beta} - g^{\beta i} V_r \begin{Bmatrix} r \\ \alpha\, i \end{Bmatrix} \quad . \end{aligned}$$

Equations 3) and 8) are not valid for coordinate systems in general, but only for very specially selected ones. To see this, let us expand the expression found in equation 3) by the chain rule of partial differentiation:

12) $$\frac{\partial Q}{\partial x^i} = \frac{\partial Q}{\partial x_j}\frac{\partial x_j}{\partial x^i} \quad .$$

Comparing equations 3) and 12) we note that equation 3) is valid only for those systems in which the relationship between the covariant and contravariant components is given by

13) $$g_{ji} = \frac{\partial x_j}{\partial x^i} \quad .$$

((There is, of course, no assurance that the system of partial differential equations represented by 13) can be solved.))

It should not be inferred that any set of covariant variables whose expression is found by solving equations 13) are covariant

coordinates. Thus, while a relationship such as 13) will guarantee that 3) is a legitimate relationship, it gives no assurance that the covariant variables involved are coordinates. Let us therefore investigate the conditions under which the covariant variables will indeed be coordinates.

Consider the basic relationship

1) $$dx_i = g_{ij}\, dx^j$$

which relates the contravariant component dx^j with its associate dx_i. We assume that the dx^j are infinitesimal displacements - in other words, coordinates. For the associates also to be coordinates, it is necessary and sufficient that equation 1) be an exact differential expression. If it is, the value of dx_i is independent of the path between the end points of the interval dx_i. Thus the dx_i are point functions.

On the other hand, if 1) is not exact the equation states that the dx_i are linear combinations of vectors. They are therefore vectors, of course, but they are not coordinates.

The condition that 1) be exact is the standard one:

14) $$\frac{\partial g_{im}}{\partial x^n} = \frac{\partial g_{in}}{\partial x^m} .$$

If the components of the metric tensor of the contravariant coordinates satisfy 14), then the set of covariant components resulting from the solution of equations 13) are coordinates, and the expressions such as $\frac{\partial Q}{\partial x_\beta}$ and $\frac{\partial V_\alpha}{\partial x_j}$ are derivatives with respect to a displacement.

As an example, suppose we choose the coordinate system related to Cartesian coordinates by

15)
$$\begin{cases} x = \xi + \eta\cos\theta \\ y = \quad\;\; \eta\sin\theta \end{cases} .$$

We have

16)
$$\begin{cases} \dfrac{\partial x}{\partial \xi} = 1 & \dfrac{\partial y}{\partial \xi} = 0 \\ \dfrac{\partial x}{\partial \eta} = \cos\theta & \dfrac{\partial y}{\partial \eta} = \sin\theta \end{cases} .$$

Thus

17)
$$\begin{cases} g_{11} = \left(\dfrac{\partial x}{\partial \xi}\right)^2 + \left(\dfrac{\partial y}{\partial \xi}\right)^2 = 1 \\ g_{12} = g_{21} = \dfrac{\partial x}{\partial \xi}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial y}{\partial \xi}\dfrac{\partial y}{\partial \eta} = \cos\theta \\ g_{22} = \left(\dfrac{\partial x}{\partial \eta}\right)^2 + \left(\dfrac{\partial y}{\partial \eta}\right)^2 = 1 \end{cases} .$$

Then

18)
$$\begin{cases} \dfrac{\partial g_{11}}{\partial \eta} = 0 & \dfrac{\partial g_{21}}{\partial \eta} = 0 \\ \dfrac{\partial g_{12}}{\partial \xi} = 0 & \dfrac{\partial g_{22}}{\partial \xi} = 0 \end{cases} .$$

In other words,

19)
$$\begin{cases} \dfrac{\partial g_{11}}{\partial x^2} = \dfrac{\partial g_{12}}{\partial x^1} \\ \dfrac{\partial g_{21}}{\partial x^2} = \dfrac{\partial g_{22}}{\partial x^1} \end{cases}$$

so that in this system the covariant components are coordinates, which we can find explicitly if we can solve 13):

13)
$$\frac{\partial x_j}{\partial x^i} = g_{ji} .$$

Let us call the related covariant coordinates ψ and ζ .

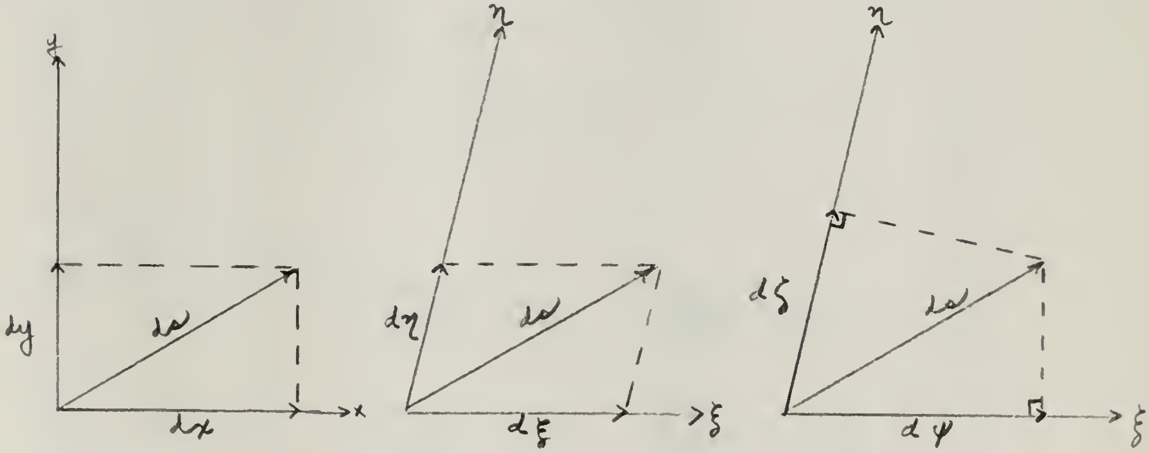
Then from 13) and 17) we have

20)
$$\begin{cases} \dfrac{\partial \psi}{\partial \xi} = 1 \\ \dfrac{\partial \psi}{\partial \eta} = \dfrac{\partial \zeta}{\partial \xi} = \cos\theta \\ \dfrac{\partial \zeta}{\partial \eta} = 1 \end{cases}$$

As can readily be checked, a solution of these equations is

21)
$$\begin{cases} \psi = \xi + \eta \cos\theta \\ \zeta = \eta + \xi \cos\theta \end{cases}$$

The relationships obtaining between Cartesian coordinates and the contravariant and covariant coordinates can be more readily understood by referring to the figure below, in which a displacement ds is referred to three sets of axes:



Another example is given by the relationship

22) $$\begin{cases} x = \xi \eta \\ y = \frac{1}{2}(\xi^2 + \eta^2) \end{cases} ,$$

although here we must restrict our investigation to the wedge shaped segment of the first quadrant lying between the forty-five and ninety degree lines, in order to assure that ξ and η will be real and single-valued.

We have

23) $$\begin{cases} \frac{\partial x}{\partial \xi} = \eta & \frac{\partial y}{\partial \xi} = \xi \\ \frac{\partial x}{\partial \eta} = \xi & \frac{\partial y}{\partial \eta} = \eta \end{cases} .$$

Thus

24) $$\begin{cases} g_{11} = \xi^2 + \eta^2 \\ g_{12} = g_{21} = 2\xi\eta \\ g_{22} = \xi^2 + \eta^2 \end{cases} .$$

Then

25) $$\begin{cases} \frac{\partial g_{11}}{\partial \eta} = 2\eta & \frac{\partial g_{21}}{\partial \eta} = 2\xi \\ \frac{\partial g_{12}}{\partial \xi} = 2\eta & \frac{\partial g_{22}}{\partial \xi} = 2\xi \end{cases} .$$

In other words,

26) $$\begin{cases} \frac{\partial g_{11}}{\partial x^2} = \frac{\partial g_{12}}{\partial x^1} \\ \frac{\partial g_{21}}{\partial x^2} = \frac{\partial g_{22}}{\partial x^1} \end{cases} ,$$

so that in this system the covariant components are coordinates also. Calling these new covariant coordinates u and ν in order to avoid confusion with the previous example, we have

27) $$\begin{cases} \dfrac{\partial u}{\partial \xi} = \xi^2 + \eta^2 \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial \nu}{\partial \xi} = 2\xi\eta \\ \dfrac{\partial \nu}{\partial \eta} = \xi^2 + \eta^2 \end{cases}$$

A solution of these equations is

28) $$\begin{cases} u = \xi\eta^2 + \frac{1}{3}\xi^3 \\ \nu = \xi^2\eta + \frac{1}{3}\eta^3 \end{cases}$$

It is possible to simplify condition 14) for orthogonal systems. Let us return to equation 14)

14) $$\frac{\partial g_{im}}{\partial x^n} = \frac{\partial g_{in}}{\partial x^m}$$

and write it as

29) $$\frac{\partial g_{ii}}{\partial x^n} = \frac{\partial g_{in}}{\partial x^i},$$

where n ≠ i.

Now for orthogonal systems,

30) $$g_{in} = 0 \qquad n \neq i,$$

so that

31) $$\frac{\partial g_{in}}{\partial x^i} = 0 \qquad n \neq i.$$

Thus, 29) becomes

32) $$\frac{\partial g_{ii}}{\partial x^m} = 0 \qquad m \neq i \, .$$

But, since this is true for all $m \neq i$ we must have

33) $$g_{ii} = f_i(x^i) \qquad \text{no sum on } i \, .$$

Thus the orthogonal systems in which expressions such as $\frac{\partial Q}{\partial x_\beta}$ and $\frac{\partial V_\alpha}{\partial x_j}$ represent derivatives with respect to covariant coordinates are only those for which 33) holds. Since each g must be a function of a single coordinates, the only one of the common systems satisfying 33) is the Cartesian system. It should be noted, however, that in this system 33) does not exclude the possibility of different scales along the various axes.

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